Probing Topological Superconductors with Sound Waves

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Abstract

A new method is introduced for probing Topological Superconductors.

The integration of the superconding fermions generates a topological **Chern** – **Simons** sound action .

Dislocations induce Majorana zero modes inside the sample, resulting in a new Hamiltonian which couple the Majorana modes, the electron field and the non-Abelian strain field sound. This Hamiltonian is used to compute the anomalous sound absorption. The Topological superconductor absorbs sound below the superconducting gap due to the transition between the quasi particles and the Majorana fermions.

The sound waves offers a new tool for detecting Majorana fermions.

I-Introduction

Topological materials have been discovered recently [1, 2]. A new class of insulators coined Topological Insulators are characterized by the second **Chern** number [15] which can be measured using the Faraday and Kerr rotation [2]. Similarly to Topological Insulators, Topological Superconductors have been found and identified by the presence of the Majorana zero modes [4–7, 12]. An indirect observation for the Majorana fermions was obtained from the differential conductance measurement [8]. Since the electrical current is not conserved in a superconductor, the electromagnetic field can not be used to identify the Topological Superconductors. The response of a Topological Superconductor is characterized by a topological $\mathbf{Chern-Simons}$ sound action as we have for the Yang-Mills action [3].The coeficient of this action is the central charge **c** which counts the number of the Majorana edge which are the signature for the Topological Superconductor. The topological action contains high derivatives and it is unlikely to be obseved in the laboratory. For this reason we need to find an alternative method.

Sound waves have played an important role in the field of Superconductivity. The first experimental measurement of the superconducting gap was by ultrasound attenuation [16]. A typical ultrasonic attenuation wave has a frequency of $10-100MHZ$ which is too small to break a pair and excite a quasi-particle, since gaps are in the range of milli-electron volts. This suggests that in order to enhance the sound absorption one needs to increase the density of the normal matter ρ_n in the superconductor. This can be done with the help of vortices or dislocations.

In a superfluid it has been shown that in the mixture ^4He-^3He, the density of the normal matter ρ_n increases with ^3He [19, 20]. The technique to measure the normal density ρ_n is based on the torsional oscillator [18], the period of the oscillation is related to the moment of inertia of the normal density. Similar measurements have been done by [17] to investigate $^3H_e-B$.

Here we would like to propose a new method based on the coupling of sound waves to Majorana fermions and electrons. We will present a theory which demonstrates that the presence of the induced Majorana modes affects the response function . We compute the sound **polarization** and obtain the normalized sound wave equation which is used to evaluate the shear and longitudinal stress. The stress serves as a probe for the Majorana

fermions. The stress field generates dislocations and strain fields. The effect of the stress field on the superconductor can be measured using impedance techniques, by measuring the change of the quality factor Q and resonance frequency of an ac-cut quartz transducer which oscillates in a shear or longitudinal mode

Topological Superconductors are characterized by a winding number and the central charge **c** in the the region where the chemical potential is positive. At the interface between the region of positive chemical potential and a negative one, (where the superconductor is not a topological superconductor) zero modes (half vortices) appear [12–14]. Half vortices are identified with the Majorana zero modes. The formation of the Majorana fermion is due to the vanishing of the Topological Superconducting density, which on a two dimensional surface is identified with the vortices. Solids are characterized by dislocations which are either induced by large external deformations or are a result of the crystal growing process. On a surface, the presence of the dislocations are similar to two dimensional vortices. The Majorana zero modes are bound to the vortices on the surface of a Topological Superconductor. This suggests that dislocations control the density of the Majorana fermions. By increasing the density of the vortices in a Superconductor, one can enhance the acoustic absorption. We find an anomalous absorption for frequencies $\hat{\Delta} - \epsilon_a < \omega < \hat{\Delta} + \epsilon_a$, $\hat{\Delta}$ is the superconducting gap and ϵ_a represents the overlapping energies for two Majorana fermions. The anomalous absorption occurs in the forbidden superconducting frequency region $\omega < 2\hat{\Delta}$. The anomalous absorption represents the finger print of the Majorana fermions in a Topological Superconductor.

This results are obtained with help of a new Hamiltonian. Using a space covariant transformation induced by the sound waves we derive the coupling between the non-Abelian strain field, the Majorana fermion and the electron field. The sound waves "deforms" the Topological Superconductor. The deformation is obtained with the help of the coordinates transformation method [21–23].

The plan of this paper is as follows . In Sec. II we introduce the Topological Superconductor Hamiltonian. Using the $p - wave$ model we construct the deformation caused by elasticity, following the methods given in the literature [15, 21–23]. We show that the integration of the fermions generate the topological **Chern − Simons** term with the central charge which counts the Majorana modes on the boundary of the sample. In the second step we consider dislocations which induce Majorana zero modes inside the sample. For

this situation we obtain a new Hamiltonian and show that the Majorana modes couple to the electron field and sound. The details are given in **Appendix -A** and **Appendix -B**. Sec. III is devoted to the computation of the sound absorption, the details are given in **Appendix -C**. We compute the shear and longitudinal stress and identify the anomalous sound absorption which confirms the presence of the Majorana modes. In section IV we present our conclusions.

II- The Topological Superconductor in the presence of an elastic strain field

In order to demonstrate the anomalous absorption we consider a $p-wave$ superconductor [4, 7]. The $p-wave$ superconductor is engineered using materials with strong spin orbit interaction and magnetic Zeeman fieldsi which are in proximity to an s-wave superconductors [12, 13] . In order to study the stress response and absorption we need to know the coupling between the sound waves and the superconductor. The form of the electron phonon Hamiltonian is problematic for spinors. The symmetry of the coupling is obtained from the coordinate transformation method [15]. When a sound waves excites the crystal, the crystal coordinates are modified. The external strain field modifies the position of the crystal atom. As a result the $p-wave$ is replaced by a **deformed** superconductor Hamiltonian:

$$H = H^{(p-wave)} + \delta H^{(cr-p-wave)} + H^{(cr)} + H^{(ext.)} \tag{1}$$

$H^{(p-wave)}$ is the model for the $p-wave$ superconductor in the absence of the sound waves.

$$\begin{aligned} H^{(p-wave)} = \frac{1}{2}\int d^2rC^{\dagger}(\vec{r},t)\Big[\tau_3\Big(\frac{\hbar^2}{2m}(-i\vec{\partial}_r)^2 - \mu_F(\vec{r})\Big) - \Delta(\vec{r},t)\Big(\tau^1 - i\tau^2\Big)(\partial_1 + i\partial_2) \\ +\Delta^*(\vec{r},t)\Big(\tau^1 + i\tau^2\Big)(\partial_1 - i\partial_2)\Big]C(\vec{r},t), \end{aligned} \tag{2}$$

$\Delta(\vec{r},t)$ the pairing order field, $\mu_F(\vec{r})$ is the space dependent chemical potential and τ^1,τ^2 τ^3 are the Pauli matrices in the particle-hole space. We assume that in one region $\mu_F(\vec{r}) > 0$ and in the complimentary region $\mu_F(\vec{r}) < 0$. For $\mu_F(\vec{r}) > 0$ the superconductor is topological and is characterized by the topological invariant with the **Chern** number $\mathbf{Q}\in\mathbf{Z}$ [12], in the region $\mu_F(\vec{r}) < 0$ the superconductor is non topological. At the interface $\mu_F(\vec{r}) = 0$ (between the two regions) the spectrum will contain bound states, Majorana zero modes. The change of sign of the chemical potential in space gives rise to the Majorana fermions.

In the literature a few proposal exist, for the Majorana fermions, mostly at the interface with a Ferromagnetic region [12] and vortices [11]. Recently new proposals based on the interplay of magnetism and superconductivity [9, 10] have been introduced. The change of sign of the chemical potential in a $p-wave$ superconductor can be achieved by the presence of vortices or dislocations (introduced by the growing process of the crystal) or large mechanical stresses. The presence of the dislocations localized at position $\vec{R}_b$ gives rise to discrete points where the chemical potential $\mu_F(\vec{r}\approx\vec{R}_b)$ vanishes (see **Appendix-A**).

The crystal Hamiltonian controls the sound propagation. The crystal Hamiltonian $H^{(cr)}$ is given by:

$$H^{(cr)}=\int d^2r\Big[\frac{\vec{\pi}^2}{2\rho}+\frac{\mu}{2}\Big(\partial_iu^j+\partial_ju^i\Big)^2+\frac{\lambda}{4}(\partial_iu^i)^2\Big];i,j=1,2 \tag{3}$$

The solid Hamiltonian $H^{(cr)}$ contains the kinetic and potential energy. The kinetic energy $\frac{\vec{\pi}^2}{2\rho}$ is determined by the canonical momentum $\vec{\pi}$ and mass density ρ. The potential energy is characterized by the *Lame* elastic constant λ and shear modulus μ [21].
$\vec{f}_{ext.}(\vec{q},t)=\vec{f}_{ex.}(\vec{q})e^{i\omega t}$ is the force of an $A-C$ quartz transducer which pumps energy into the solid. The external source Hamiltonian $H^{(ext.)}$ is given by:

$$H^{(ext.)}=\int d^2r\vec{f}_{ext.}(\vec{r},t)\cdot\vec{u}(\vec{r},t) \tag{4}$$

The sound propagation modifies the $p-wave$ .In the presence of the sound waves, the coordinates are modified from $\vec{r}\equiv\vec{x}=[x^{(1},x^{(2)}]$ to $\vec{x}+\vec{u}(\vec{x},t)=\vec{X}=\Big[X^{a=1},X^{a=2}\Big]$ where $\vec{u}(\vec{x},t)=[u^{(1)}(\vec{x},t),u^{(2)}(\vec{x},t)]$. The sound modifies the term $\tau^a\partial_a$ (in the Hamiltonian Eq.(2)) to $\sum_{a=1,2}\tau^a\frac{\partial x^{(i)}}{\partial X^{(a)}}\nabla_i$. $\frac{\partial x^{(i)}}{\partial X^{(a)}}$ is computed with the help of the transformation $\vec{X}(\vec{x})$ induced by the sound. As a result a non-Abelian strain field $\vec{A}(\vec{x},t)$ is generated. The non-Abelian field $\vec{A}(\vec{x},t)$ is written in terms of the elastic strain field.The non-Abelian form is due to the particle-hole operators τ^1,τ^2 τ^3 represents the effect of the sound waves which is written in terms of the elastic field $\vec{u}(\vec{x},t)$. The full derivation of $H^{(deformed-p-wave)}$ Hamiltonian is given in **Apendix -B**. The integration of the fermions with Hamiltonian $H^{(deformed-p-wave)}$ and $\mu_F>0$ generate a topological sound action,

$$S^{top-sound}=\frac{c}{96\pi}\int dt\int d^2r\Big[\epsilon^{i,j,k}\omega_{i,a}(\partial_j\omega^a_k-\partial_k\omega^a_j)+\frac{2}{3}\epsilon^{a,b,c}\omega_{i,a}\omega_{i,b}\omega_{i,c}\Big] \tag{5}$$

c counts the number of the Majorana modes on the edge of the sample for $\mu_F>0$. Due to the high order derivatives $S^{top-sound}$ is unlikely to be observed in the laboratory.

We will consider a chemical potential which changes sign inside the sample due to vortices or dislocations. Due to vortices or dislocations we will have additional Majorana zero modes. We will focus on the first non-trivial $\delta H^{(cr-p-wave)} \equiv H^{(deformed-p-wave)} - H^{(p-wave)}$ term which contributes to the absorption. Such a term is linear in the Majorana and in the electron field.

$$
\begin{aligned}
&H^{(deformed-p-wave)} - H^{(p-wave)} \approx \delta H^{(cr-p-wave)} = \int d^2r \Big[C^{\dagger}(\vec{r},t)\Big(\vec{A}(\vec{r},t)\cdot\vec{\partial}_r\Big)C(\vec{r},t)\Big],\\
&\vec{A}(\vec{r},t) = I\vec{b}_0(\vec{r},t) + \tau_1\vec{b}_1(\vec{r},t) + \tau_2\vec{b}_2(\vec{r},t), \quad \Delta = |\Delta|e^{i\alpha}, \quad \vec{b}_0(\vec{r},t) = \partial_t\vec{u}(\vec{r},t)\\
&\vec{b}_1(\vec{r},t) = i|\Delta|\Big(\cos(\alpha)\partial_2\vec{u}(\vec{r},t) - \sin(\alpha)\partial_1\vec{u}(\vec{r},t)\Big) \;\; \vec{b}_2(\vec{r},t) = i|\Delta|\Big(\cos(\alpha)\partial_1\vec{u}(\vec{r},t) + \sin(\alpha)\partial_2\vec{u}(\vec{r},t)\Big)
\end{aligned}
\tag{6}
$$

The $H^{(p-wave)}$ Hamiltonian is diagonalized with the help of the Bogoliubov - deGennes transformation. The Hamiltonian has in addition to the positive eigenvalues also Majorana zero modes [4, 7]. The spinor for non-zero energies are given by $\Big[U(\vec{k}),V(\vec{k})\Big]^T$ and the zero modes spinor are given by $\Big[U_b(\vec{r},\phi),U^*_b(\vec{r},\phi)\Big]^T$. The $Bogoliubov - deGennes$ Hamiltonian contains pairs of momentum $[\vec{k},-\vec{k}]$ and therefore the momentum integration is restricted to half of the Brillouin zone. To cover the entire Brillouin Zone we will replace $\Big[U(\vec{k}),V(\vec{k})\Big]^T$ by a four component spinor $\Phi(\vec{k}) = \Big[U(\vec{k}),V(\vec{k}),U(-\vec{k}),V(-\vec{k})\Big]^T$, similarly for the Majorana spinor we will replace $\Big[U_b(\vec{r},\phi),U^*_b(\vec{r},\phi)\Big]^T$ with a four component spinor $\hat{W}_b(\vec{r},\phi) = \Big[U_b(\vec{r},\phi),U^*_b(\vec{r},\phi),U_b(\vec{r},\phi),U^*_b(\vec{r},\phi)\Big]^T$.

$$
\begin{aligned}
\mathbf{C}(\vec{r}) &= \int \frac{d^2k}{(2\pi)^2}e^{i\vec{k}\vec{r}}\Big[\eta(\vec{k})\Phi_(\vec{k}) + \eta(-\vec{k})\Gamma\Phi^*(\vec{k})\Big] + \sum_{b=1}^{2n}\gamma_b\hat{W}_b(\vec{r},\phi)\\
\mathbf{C}^{\dagger}(\vec{r}) &= \int \frac{d^2k}{(2\pi)^2}e^{-i\vec{k}\vec{r}}\Big[\eta^{\dagger}(\vec{k})[\Phi^*(\vec{k})]^T + \eta(-\vec{k})[\Phi(-\vec{k})]^T\Gamma\Big] + \sum_{b=1}^{2n}\gamma_b[\hat{W}^*_b(\vec{r},\phi)]^T
\end{aligned}
\tag{7}
$$

$\mathbf{C}(\vec{r})$ and $\mathbf{C}^{\dagger}(\vec{r})$ are the quasi particles operator for the superconductor. The matrix Γ ensures the (pseudo) reality conditions [11].

$$
\Gamma\mathbf{C}^{\dagger}(\vec{r}) = \mathbf{C}(\vec{r}), \quad \Gamma\Phi^*_E(\vec{k}) = \Phi_{-E}(\vec{k}) \tag{8}
$$

The operators $\mathbf{C}(\vec{r})$, $\mathbf{C}^{\dagger}(\vec{r})$ also contain the **Majorana zero modes** $\sum_{b=1}^{2n}\gamma_b\hat{W}_b(\vec{r},\phi)$, $\sum_{b=1}^{2n}\gamma_b[\hat{W}^*_b(\vec{r},\phi)]^T$ where γ_b are the Majorana operators. Using the mode expansion given

in Eq.(7) we will replace the Hamiltonian $\delta H^{(cr-p-wave)}$ by:

$$\delta H^{(cr-p-wave)} = \int d^2r \int \frac{d^2k}{(2\pi)^2} e^{-i\vec{k}\vec{r}} \Big[\eta^\dagger(\vec{k})[\Phi^*(\vec{k})]^T + \eta(-\vec{k})[\Phi(-\vec{k})]^T\Gamma\Big]\Big(\vec{A}(\vec{r},t)\cdot\vec{\partial}\Big)\sum_{b=1}^{2n}\gamma_b[\hat{W}_b^*(\vec{r},\phi)]^T\Big] + h.c. \tag{9}$$

III- Computation of the anomalous sound absorption

In order to perform the computation in Eq.(9) we need to know the spatial wave function for the Majorana fermions. Due to their localization in space and low density we can assume a model of randomly distributed Majorana fermions. We will consider a situation where the correlation within the nearest neighbor pair $\Big[\gamma_{b=2a-1},\gamma_{b'=2a}\Big]$ is significant and negligible otherwise. This allows to introduce the fermion operators $\zeta_a^\dagger$,ζ_a ;

$$\gamma_{2a-1} = \frac{1}{\sqrt{2}}\Big[\zeta_a^\dagger + \zeta_a\Big], \quad \gamma_{2a} = \frac{1}{i\sqrt{2}}\Big[\zeta_a^\dagger - \zeta_a\Big], a = 1.2..n \tag{10}$$

The overlap between the pair $\Big[\gamma_{b=2a-1},\gamma_{b'=2a}\Big]$ is given by the overlapping energy ϵ_a. The Majorana Hamiltonian is given by $H^{Majorana} = \sum_{a=1}^{a=n}\epsilon_a\zeta_a^\dagger\zeta_a$. The ground state with the Majorana fermions is given by $|G,\epsilon_{a=1}...\epsilon_{a=n}\rangle$. This ground state $|G,\epsilon_{a=1}...\epsilon_{a=n}\rangle$ annihilates the operators $\eta(\vec{k})$ and ζ_a, $\eta(\vec{k})|G,\epsilon_{a=1}...\epsilon_{a=n}\rangle = 0$, $\zeta_a|G,\epsilon_{a=1}...\epsilon_{a=n}\rangle = 0$. We compute the expectation value with respect the ground state $|G,\epsilon_{a=1}...\epsilon_{a=n}\rangle$ and find the sound **polarization** $\mathbf{\Pi}[\vec{q},\omega]$.

$$\langle G,\epsilon_{a=1}...\epsilon_{a=n}|Te^{\frac{-i}{\hbar}\int_{-\infty}^{\infty}dt\delta H^{(cr-p-wave)}}|G,\epsilon_{a=1}...\epsilon_{a=n}\rangle \approx Te^{\frac{-i}{\hbar}\int_{-\infty}^{\infty}dt\delta H^{(cr)}[\partial_i u^j(\vec{r},t),\partial_t u^i(\vec{r},t)]} \tag{11}$$

As a result the sound wave Hamiltonian is replaced by: $H = H^{(cr)} + H^{(ext.)} + \delta H^{(cr)}$ where $\delta H^{(cr)}$ is given by Eq.(11). (It gives the sound absorption induced by the transition between the superconductor quasi-particles with energy $E = \hbar v\sqrt{(\hat{\epsilon}(\vec{k}) - \hat{\mu}_F)^2 + |\Delta|^2k^2}$ and the Majorana fermions ϵ_a.) We evaluate Eq.(11) to order $\frac{1}{\hbar^2}$ and obtain the polarization diagram $\mathbf{\Pi}[\vec{q},\omega]$.

$$\delta H^{(cr)} = \int\frac{d^2k}{(2\pi)^2}\int\frac{d\omega}{2\pi}\mathbf{\Pi}[\vec{q},\omega]\vec{u}(\vec{q},\omega)\cdot\vec{u}(-\vec{q},-\omega), \quad \mathbf{\Pi}[\vec{q},\omega] \equiv \mathbf{Re\Pi}[\vec{q},\omega] + \mathbf{Im\Pi}[\vec{q},\omega] \tag{12}$$

The polarization $\mathbf{\Pi}[\vec{q},\omega]$ is computed in **Appendix-C**. Using typical values for the sound waves frequency $\omega = 10^9Hz$ and Fermi momentum $k_F = 10^9m^{-1}$ we find that the value of

the self energy $\mathbf{\Pi}[\vec{q},\omega]$ is in the range $100\frac{Newton}{m^2}$, therefore the effect on the phonon frequency is negligible, and we will consider only the imaginary part of the polarization $\mathbf{\Pi}[\vec{q},\omega]$ which determines the absorption.

$$\mathbf{\Pi}[\vec{q},\omega] = \frac{\hbar\omega^2 k_F^3}{8\pi}\Big[1-2|\Delta|\frac{|\vec{q}|}{|\omega|}\cos[\alpha-\beta(\vec{q})]-4|\Delta|^2(\frac{|\vec{q}|}{|\omega|})^2\Big]\cdot\sum_{a=1}^{n}$$
$$\int\frac{d\hat{E}}{\sqrt{\hat{E}^2-\Delta^2k_F^2}}\frac{(\sqrt{\hat{E}^2-\Delta^2k_F^2}-\hat{E})^2}{(\sqrt{\hat{E}^2-\Delta^2k_F^2}-\hat{E})^2+\Delta^2k_F^2}\Theta[\hat{E}-\Delta k_F]\cdot$$
$$\Big[\Theta[\omega]\Big((\Theta[\hat{E}]\Theta[\hat{\epsilon}_a]-\Theta[-\hat{E}]\Theta[-\hat{\epsilon}_a])\delta[\omega-(\hat{E}+\hat{\epsilon}_a)]+(\Theta[-\hat{E}]\Theta[\hat{\epsilon}_a]+\Theta[\hat{E}]\Theta[-\hat{\epsilon}_a])\delta[\omega-(\hat{E}-\hat{\epsilon}_a)]\Big)+$$
$$\Big[\Theta[-\omega]\Big((\Theta[\hat{E}]\Theta[\hat{\epsilon}_a]-\Theta[-\hat{E}]\Theta[-\hat{\epsilon}_a])\delta[\omega+(\hat{E}+\hat{\epsilon}_a)]+(\Theta[\hat{E}]\Theta[-\hat{\epsilon}_a]+\Theta[\hat{E}]\Theta[-\hat{\epsilon}_a])\delta[\omega+(\hat{E}-\hat{\epsilon}_a)]\Big)\Big] \quad (13)$$

In Eq.(12) we used the definitions $\epsilon(\vec{k})=\frac{\hbar^2}{2m}\vec{k}^2$, $\hat{\epsilon}(\vec{k})=\frac{\epsilon(\vec{k})}{\hbar v}$,$\hat{\mu}_F=\frac{\mu_F}{\hbar v}$, $\hat{E}=\frac{E}{\hbar v}=\sqrt{(\hat{\epsilon}(\vec{k})-\hat{\mu}_F)^2+|\Delta|^2k^2}$.

The first line of Eq.(13) is determined by the structure of the sound field $\vec{A}(\vec{r},t)$ given in Eq.(3). $\sum_{a=1}^{n}$ represents the sum over the uncorrelated Majorana fermions ϵ_a.The second line of Eq.(13) contains the density of states $\frac{\Theta[\hat{E}-\Delta k_F]}{\sqrt{\hat{E}^2-\Delta^2k_F^2}}$ and the coherence factor $\frac{(\sqrt{\hat{E}^2-\Delta^2k_F^2}-\hat{E})^2}{2((\sqrt{\hat{E}^2-\Delta^2k_F^2}-\hat{E})^2+\Delta^2k_F^2)}$ (introduced by the spinors $\Phi(\vec{k})$, $\hat{W}_a(\vec{r},\phi)$)

The **third** line with $\Theta[\omega]$ $(\omega>0)$ describes the sound absorption.

The **first** term in the the third line describes the creation of a Majorana hole with the energy ϵ_a (destruction of Majorana particle with the energy $-\epsilon_a$) and the creation of a electron with energy E. As a result the absorption of sound will occur for frequencies $\omega>\hat{E}+\hat{\epsilon}_a$ with the threshold absorption $\omega>|\Delta|k_F+\hat{\epsilon}_a$. In the absence of the Majorana term the absorption will occur $\omega>2\hat{E}$,resulting in a forbidden frequency region $\omega>2|\Delta|k_F$.

The **second** term in the third line describes a situation where the Majorana states are occupied at the energy $\hat{\epsilon}_a$, and the sound absorption will excite the electron to the state energy $\hat{E}$. As a result the sound will be absorbed for frequencies $\omega>\hat{E}-\hat{\epsilon}_a$,resulting in the threshold absorption $\omega>|\Delta|k_F-\hat{\epsilon}_a$. This absorption will be controlled by the Fermi Dirac occupation function which at finite temperatures T replaces $\Theta[-\epsilon_a]$ with $n_F[\epsilon_a]=\frac{1}{e^{\frac{\epsilon_a}{K_BT}}+1}$. The absorption region $|\Delta|k_F-\hat{\epsilon}_a<\omega<|\Delta|k_F+\hat{\epsilon}_a$ will be called the **anomalous absorption**. Due to the fact that $|\Delta|k_F>>\hat{\epsilon}_a$ the presence of the Fermi Dirac function $n_F[\epsilon_a]$ in the absorption formula gives the possibility to determine the energy distributions

of the Majorana fermions.

The **fourth** line with $\Theta[-\omega]$ ($\omega<0$) describes the sound emission.

We will use the polarization operator $\mathbf{\Pi}[\vec{q},\omega]$ obtained in Eq.(13) to compute the sound waves propagation in the crystal. We note that due to the random distribution of the Majorana fermions the sound polarization is isotropic and will affect the sound waves equations in a symmetric way (see Eq.(14)). The crystal Hamiltonian is normalized by the sound polarization $\mathbf{\Pi}[\vec{q},\omega]\vec{u}(\vec{q},\omega)\cdot\vec{u}(-\vec{q},-\omega)$. The sound wave equation for the external force $f^{(i)}(\vec{q},\omega)$ is given by:

$$\Big[(-\omega^2+\mu q^2+\mathbf{\Pi}[\vec{q},\omega])\delta_{i,j}+(\lambda+\mu+\frac{\mathbf{\Pi}[\vec{q},\omega]}{q^2})q_iq_j\Big]u^{(j)}(\vec{q},\omega)=f^{(i)}(\vec{q},\omega);\quad i,j=1,2 \tag{14}$$

We consider a situation where the force in the x direction is zero $f^{(1)}(\vec{q},\omega)=0$, and the force in the y direction is $f^{(2)}(\vec{q},\omega)\neq 0$. We compute the **shear stress** $\sigma_{\mathbf{1},\mathbf{2}}[\vec{q},\omega]$, the **longitudinal stress** $\sigma_{\mathbf{2},\mathbf{2}}[\vec{q},\omega]$, the **strain tensor** $\epsilon_{1,2}[\vec{q},\omega]$, $\epsilon_{2,2}[\vec{q},\omega]$ the impedance $\mathbf{Z}[\vec{q},\omega]$ in terms of the **shear modulus** μ and **Lame** elastic constant λ .Using the solutions obtained from Eq.(14) we find:

$$\begin{aligned}
&\sigma_{\mathbf{1},\mathbf{2}}[\vec{q},\omega]=\epsilon_{1,2}[\vec{q},\omega]2\mu,\quad \epsilon_{1,2}[\vec{q},\omega]\equiv\frac{1}{2}\Big[iq_2u^{(1)}(\vec{q},\omega)+iq_1u^{(2)}(\vec{q},\omega)\Big]\\
&\sigma_{\mathbf{2},\mathbf{2}}[\vec{q},\omega]=\epsilon_{2,2}[\vec{q},\omega](3\lambda+2\mu),\quad \epsilon_{2,2}[\vec{q},\omega]\equiv\Big[iq_2u^{(2)}(\vec{q},\omega)\Big]\\
&\mathbf{Z}_{1,2}[\vec{q},\omega]=\frac{\sigma_{\mathbf{1},\mathbf{2}}[\vec{q},\omega]}{f^{(2)}[\vec{q},\omega]}\equiv\mathbf{R}_{1,2}[\vec{q},\omega]+i\mathbf{X}_{1,2}[\vec{q},\omega]=\frac{i2\mu\frac{q_1^3}{q^2}}{\mu q^2+i\mathbf{Im}\mathbf{\Pi}[\vec{q},\omega]}\\
&\mathbf{Z}_{2,2}[\vec{q},\omega]=\frac{\sigma_{\mathbf{2},\mathbf{2}}[\vec{q},\omega]}{f^{(2)}[\vec{q},\omega]}\equiv\mathbf{R}_{2,2}[\vec{q},\omega]+i\mathbf{X}_{2,2}[\vec{q},\omega]=\frac{i(3\lambda+2\mu)q_2}{\mu q^2+i\mathbf{Im}\mathbf{\Pi}[\vec{q},\omega]}
\end{aligned} \tag{15}$$

We plot the real part of the impedance $\mathbf{R}_{1,2}[\vec{q},\omega]$ for a fixed Majorana energy, ignoring the energy dispersion of the Majorana fermions. Due to the isotropic form of the polarization the longitudinal impedance $\mathbf{R}_{2,2}[\vec{q},\omega]$ has the same frequency dependence as $\mathbf{R}_{1,2}[\vec{q},\omega]$ (see Eq.(14)). We will use in the plot a narrow energies distribution $\frac{\epsilon_a}{\omega_F}\approx 0.05$.

The **thin** line gives the anomalous absorption $\frac{|\Delta|k_F-\epsilon_a}{\omega_F}<\frac{\omega}{\omega_F}<\frac{|\Delta|k_F+\epsilon_a}{\omega_F}$ for the Majorana energy $\frac{\epsilon_a}{\omega_F}\approx 0.05$ and the superconducting gap $\frac{|\Delta|k_F}{\omega_F}\approx 0.25$. $\omega_F=\frac{\mu_F}{\hbar}$ is the corresponding frequency for the Fermi energy.

The **thick** line shows the absorption in the **absence** of the Majorana fermions for frequencies $\frac{\omega}{\omega_F}<\frac{2|\mathbf{\Delta}|\mathbf{k_F}}{\omega_F}$ (The form of the gap $\Delta|\vec{k}|$ is a result of the Hamiltonian given in Eq.(2).) Since

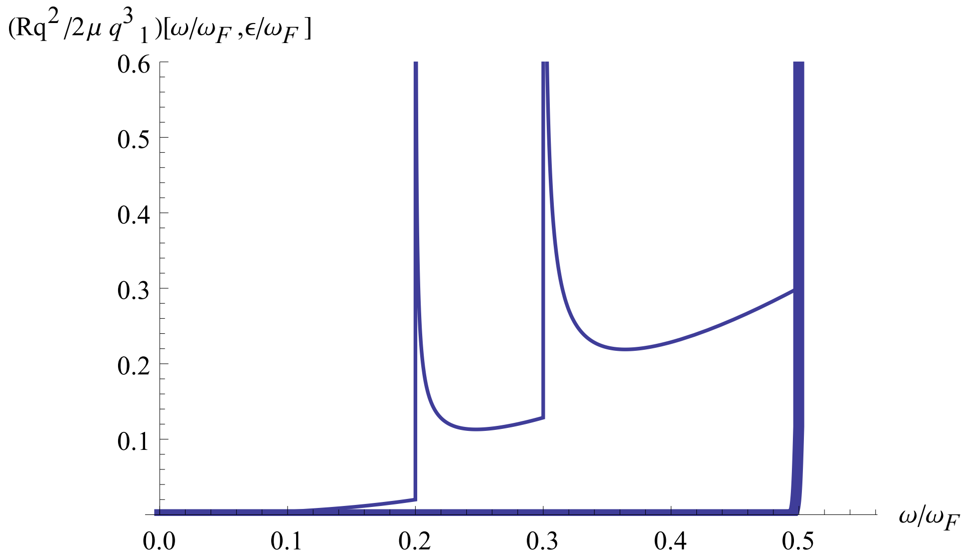

FIG. 1: The real part of the impedance is $\mathbf{R}[\vec{q},\omega]$. The **thin** line gives the anomalous absorption $\frac{|\Delta|k_F-\epsilon_a}{\omega_F} < \frac{\omega}{\omega_F} < \frac{|\Delta|k_F+\epsilon_a}{\omega_F}$ for the Majorana energy $\frac{\epsilon_a}{\omega_F} \approx 0.05$ and the superconducting gap $\frac{|\Delta|k_F}{\omega_F} \approx 0.25$. The anomalous absorption is plotted for low temperatures such that $n_F[\epsilon_a] \approx 1$. The **thick** line shows the sound absorption in the absence of the Majorana, the absorption **starts** for frequencies $\frac{\omega}{\omega_F} > \frac{2|\mathbf{\Delta}|\mathbf{k_F}}{\omega_F} = 0.5$

disorder is present, it raises the question how to differentiate the absorption between the Majorana fermions and charged impurities. Charged impurities will give rise to absorption for frequencies $\omega > \hat{E} - \hat{\epsilon}_{imp.}$ and temperatures $K_B T > 2|\Delta|k_F$, contrary to the absorption induced by the Majorana fermions at $T \to 0$.

IV-Conclusions

To conclude a new proposal for probing superconductors with sound waves has been introduced. An anomalous absorption is found in the frequency region $|\Delta|k_F - \hat{\epsilon}_a < \omega < |\Delta|k_F + \hat{\epsilon}_a$ and the energy distributions of the Majorana fermions can be determined. This gives direct evidence for the presence of the Majorana fermions. The Majorana fermions are the finger print of the Topological Superconductors. Therefore by measuring the anomalous absorption we have identified a method for detecting Majorana fermions and probing the response of the Topological Superconductors.

Appendix -A

Due to the large deformation introduced by the dislocation localized at $\vec{R}_b$, the crystal coordinates are modified from $\vec{r}=[x,y]$ to $\vec{R}(\vec{r})=[X(\vec{r}),Y(\vec{r})]$.This allows to introduce the static strain field $e_i^a=\partial_i X^a$, $a=1,2$ and $i=1,2$. Due to the dislocation the the area $dXdY$ is modified to $J[\vec{R}_b]dxdy$ where the $J[\vec{R}_b]$ is the Jacobian of the transformation, As a result the chemical potential μ_F becomes $\mu_F \to \mu_F(\vec{r}) \equiv \mu_F J[\vec{R}_b]$. For an edge dislocation the coordinate transformation is given by : $X=x$ and $Y=y-\frac{B^{(2)}}{2\pi}tan^{-1}\frac{y-R_b^{(y)}}{x-R_b^{(x)}}$. $B^{(2)}$ is the Burgers vector in the y direction. This transformation gives for the Jacobian $J[\vec{R}_b] \equiv \left(e_1^1 e_2^2 - e_2^1 e_1^2\right) = 1-\frac{B^{(2)}}{2\pi}\frac{x-R_b^{(x)}}{(x-R_b^{(x)})^2+(y-R_b^{(y)})^2}$ [21]. As a result the chemical potential $\mu_F(\vec{r}\approx\vec{R}_b)$ vanishes.

Appendix- B

The sound waves field $\vec{u}(\vec{x},t)$ change the coordinates from $\vec{r}\equiv\vec{x}=[x,y]$ to $\vec{x}+\vec{u}(\vec{x},t)=\vec{X}=\left[X^{a=1},X^{a=2}\right]$ where $\vec{u}(\vec{x},t)=[u^{(1)}(\vec{x},t),u^{(2)}(\vec{x},t)]$. The strain field are defined by $E_a^i=\partial_a x^i$, $e_i^a=\partial_i X^a$ are related. We have: $\sum_{i=1}^{2} e_i^a E_b^i=\delta_{a,b}$, $\sum_{a=1,2} e_i^a e_j^a = g_{i,j}$ and $E_{i,b}=g_{i,j}E_b^j$. For the time component we have $E_i^t=e_i^t\delta_{i,t}$.

The derivatives transform like vectors, $\partial_a=\partial_a u^1(\vec{x},t)\partial_1+\partial_a u^2(\vec{x},t)\partial_2$. For the sound waves $\vec{u}(\vec{r},t)$ we have: $e_i^1=\delta_{1,i}-\partial_i u^1(\vec{x},t)$, $e_i^2=\delta_{2,i}-\partial_i u^2(\vec{x},t)$, $e_t^a=-\partial_t u^a(\vec{x},t)$ for $a=1,2$. The integration area element d^2x in Eq.(2) is multiplied by the Jacobian $J=\left[e_1^1 e_1^2 - e_2^1 e_1^2\right] \equiv Det[e_i^a]$. The deformed $p-wave$ Hamiltonian $H^{(deformed-p-wave)}$ takes the form:

$$H^{(deformed-p-wave)}=\frac{1}{2}\int d^2x Det[e_i^a]C^\dagger(\vec{r},t)\Big[\tau^3\Big(\frac{-\hbar^2}{2m}(\sum_{a=1}^{2}E_a^i E_a^j\nabla_i\nabla_j)-\mu_F(\vec{x})\Big)-\Delta(\vec{x},t)\Big(\tau^1-i\tau^2\Big)(E_1^i$$

$$+\Delta^*(\vec{x},t)\Big(\tau^1+i\tau^2\Big)(E_1^i\nabla_i - iE_2^i\nabla_i)\Big]C(\vec{x},t)$$

∇_i is the covariant derivative given in terms of the spin connection:
$\omega_i^{a,b}[\tau^a,\tau^b]$, $\nabla_i\equiv\partial_i+\frac{1}{8}\omega_i^{a,b}[\tau^a,\tau^b]$. The spin connection has been derived in [22] in terms of E_a^i and e_i^a. The spin connection is determined from the zero torsion condition, $\nabla_i e_j^a - \nabla_j e_i^a = 0$. We have,

$$\omega_i^a=\epsilon^{a,b,c}E_c^j(\partial_i E_{j,b}-\partial_j E_{i,b})-\frac{1}{2}\epsilon^{b,c,d}(E_c^j E_c^k\partial_k E_{j,d})e_i^a \tag{17}$$

Where $E_{i,b}=g_{i,j}E_b^j$.

Once the spin connection is know we can perform the path integral over the Dirac' fermion. As for the Yang-Mills theory the fermion integration in $2+1$ dimensions generate a non-Abelian Chern-Simons term. We perform the path integral integration tor the fermion field $C^{\dagger}(\vec{r},t)$ and $C(\vec{r},t)$ and obtain the effective sound action $S^{top-sound}$ [23]. The topological term is given by,

$$S^{top-sound}=\frac{c}{96\pi}\int dt\int d^2r\Big[\epsilon^{i,j,k}\omega_{i,a}(\partial_j\omega^a_k-\partial_k\omega^a_j)+\frac{2}{3}\epsilon^{a,b,c}\omega_{i,a}\omega_{i,b}\omega_{i,c}\Big] \tag{18}$$

where $\omega_{i,c}=g_{i,j}\omega^j_c$. $\mathbf{c}$ counts the number of the edge modes. For the Topological Superconductor we have $\mu_F>0$ and $\mathbf{c}$ is non zero. For this case Majorana modes are on the edge of the sample. As a result the effective sound action $S^{top-sound}$ allows to identify the Topological Superconductor. Due too the high order derivatives it is difficult to observe such a term in the laboratory. For this reason we will look for an alternative way to identify the Topological Superconductor. This result is similar to the gravitational Chern-Simons term [3].

In the presence of dislocations we expect to have additional Majorana zero modes. Therefore the sound waves can couple to the Majorana fermions and electron field. In order to study this effect we will consider small deformations $|\vec{u}(\vec{r},t)|<<a$ (a is the lattice constant). We will keep only first order terms in the coordinate transformation and neglect the effect on the metric tensor and spin connection $\omega^{a,b}_i$.

$$H^{(deformed-p-wave)}\approx\frac{1}{2}\int d^2r\Big[C^{\dagger}(\vec{r})\Big(\frac{-\hbar^2}{2m}\delta_{i,j}\partial_i\partial_j-\mu(\vec{r})\Big)C(\vec{r})-\Delta(\vec{r},t)\Big(\tau^1-i\tau^2\Big)(E^i_1\partial_i+iE^i_2\partial_i)$$
$$+\Delta^*(\vec{r},t)\Big(\tau^1+i\tau^2\Big)(E^i_1\partial_i-iE^i_2\partial_i)\Big]C(\vec{r},t) \tag{19}$$

We define $\delta H^{(cr-p-wave)}$:

$$\delta H^{(cr-p-wave)}\equiv H^{(deformed-p-wave)}-H^{(p-wave)} \tag{20}$$

The Hamiltonian $\delta H^{(cr-p-wave)}$ is given in Eq.(3).

Appendix-C

We use Wick's theorem for the Green's functions, $G(\vec{k},\tau)=-i\langle G,\epsilon_{a=1}...\epsilon_{a=n}|T(\eta(\vec{k},\tau)\eta^{\dagger}(\vec{k},0))|G,\epsilon_{a=1}...\epsilon_{a=n}\rangle$ for the Superconductor and

$g_a(\tau) = -i\langle G, \epsilon_{a=1}...\epsilon_{a=n}|T(\zeta_a(t)\zeta_a^{\dagger}(0))|G, \epsilon_{a=1}...\epsilon_{a=n}\rangle$ for he Majorana fermions. The spinors $\Phi(\vec{k})$, $\hat{W}_a(\vec{r},\phi)$ are used to compute the "coherent" factors of the self energy $\mathbf{\Pi}[\vec{q},\omega]$:

$$\mathbf{\Pi}[\vec{q},\omega] = i\frac{\hbar\omega^2 k_F^3}{8\pi}\Big[1-2|\Delta|\frac{|\vec{q}|}{|\omega|}\cos[\alpha-\beta(\vec{q})]-4|\Delta|^2(\frac{|\vec{q}|}{|\omega|})^2\Big]\int\frac{d\hat{E}}{\sqrt{\hat{E}^2-\Delta^2k_F^2}}\cdot$$
$$\frac{(\sqrt{\hat{E}^2-\Delta^2k_F^2}-\hat{E})^2}{(\sqrt{\hat{E}^2-\Delta^2k_F^2}-\hat{E})^2+\Delta^2k_F^2}\sum_{a=1}^{n}\int d\tau\Big[\Big(\langle G,\epsilon_{a=1}...\epsilon_{a=n}|T\Big(\eta^{\dagger}(\hat{E},\tau)\eta(\hat{E},0)\Big)|G,\epsilon_{a=1}...\epsilon_{a=n}\rangle$$
$$+\langle G,\epsilon_{a=1}...\epsilon_{a=n}|T\Big(\eta(\hat{E},\tau)\eta^{\dagger}(\hat{E},0)\Big)|G,\epsilon_{a=1}...\epsilon_{a=n}\rangle\Big)\Big(\langle G,\epsilon_{a=1}...\epsilon_{a=n}|T\Big(\zeta_a^{\dagger}(\tau)\zeta_a(0)\Big)|G,\epsilon_{a=1}...\epsilon_{a=n}\rangle$$
$$+\langle G,\epsilon_{a=1}...\epsilon_{a=n}|T\Big(\zeta_a(\tau)\zeta_a^{\dagger}(0)\Big)|G,\epsilon_{a=1}...\epsilon_{a=n}\rangle\Big)\Big] \tag{21}$$

$\tau = t_1 - t_2$, T stands for the time order product. In Eq.(17) we have used the notation $\hat{E} = \frac{E}{\hbar v}$; $\hat{\epsilon}_a = \frac{\epsilon_a}{\hbar v}$; $tan[\beta(\vec{q})] = \frac{q_2}{q_1}$. We integrate with respect $\tau = t_1 - t_2$ and obtain:

$$\mathbf{\Pi}[\vec{q},\omega] = \frac{\hbar\omega^2 k_F^3}{8\pi}\Big[1-2|\Delta|\frac{|\vec{q}|}{|\omega|}\cos[\alpha-\beta(\vec{q})]-4|\Delta|^2(\frac{|\vec{q}|}{|\omega|})^2\Big]\int\frac{d\hat{E}}{\sqrt{\hat{E}^2-\Delta^2k_F^2}}\cdot$$
$$\frac{(\sqrt{\hat{E}^2-\Delta^2k_F^2}-\hat{E})^2}{(\sqrt{\hat{E}^2-\Delta^2k_F^2}-\hat{E})^2+\Delta^2k_F^2}\sum_{a=1}^{n}\Big[\frac{\Theta[\hat{E}]\Theta[\hat{\epsilon}_a]}{\omega+(\hat{E}+\hat{\epsilon}_a)-ix}+\frac{\Theta[\hat{E}]\Theta[\hat{\epsilon}_a]}{\omega-(\hat{E}+\hat{\epsilon}_a)+ix}$$
$$+\frac{\Theta[-\hat{E}]\Theta[-\hat{\epsilon}_a]}{\omega-(\hat{E}+\hat{\epsilon}_a)-ix}+\frac{\Theta[-\hat{E}]\Theta[-\hat{\epsilon}_a]}{\omega+(\hat{E}+\hat{\epsilon}_a)+ix}+\frac{\Theta[-\hat{E}]\Theta[\hat{\epsilon}_a]}{\omega+(\hat{E}-\hat{\epsilon}_a)-ix}+\frac{\Theta[-\hat{E}]\Theta[\hat{\epsilon}_a]}{\omega-(\hat{E}-\hat{\epsilon}_a)+ix}$$
$$+\frac{\Theta[\hat{E}]\Theta[-\hat{\epsilon}_a]}{\omega+(\hat{E}-\hat{\epsilon}_a)-ix}+\frac{\Theta[\hat{E}]\Theta[-\hat{\epsilon}_a]}{\omega-(\hat{E}-\hat{\epsilon}_a)+ix}\Big], x\to 0 \tag{22}$$

At finite temperatures the steps functions are replaced by the Fermi Dirac ocupation functions, $\Theta[-\hat{E}] = n_F[\hat{E}] = \frac{1}{e^{\beta\hat{E}}+1}$, $\Theta[-\hat{\epsilon}_a] = n_F[\hat{\epsilon}_a] = \frac{1}{e^{\beta\hat{\epsilon}_a}+1}$,$\Theta[\hat{E}] = 1-\Theta[-\hat{E}]$, $\Theta[\hat{\epsilon}_a] = 1-\Theta[-\hat{\epsilon}_a]$.

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